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Heat kernel expansion coefficient: II. Higher-order operators

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Abstract. In this paper we obtain useful relations for calculating the $a_{(1/2)d}$ asymptotic expansion coefficient for an associated heat kernel and elliptical operator of order greater than two in d dimensions. In our main relation we relate the $a_{(1/2)d}$ coefficients for two commuting elliptical operators and the $a_{(1/2)d}$ coefficient of their product. We use this relation to obtain results (in four dimensions) for the a_2 coefficient, including total divergences, for the fourth-order operator $(\Box^2+E^\kappa\nabla_\kappa\Box+A^{(\kappa\varepsilon)}\nabla_\kappa\nabla_\varepsilon+C^\kappa\nabla_\kappa+Z)$. We also argue that, for higher-order operators, the coefficients of the loop divergences must be slightly altered. We discuss the literature.

1. Introduction

In this paper, the second in a series of two, we discuss higher-order elliptical operators which have leading terms with more than two derivatives. We develop useful properties of the $a_{(1/2)d}$ asymptotic expansion coefficient of their associated heat kernel $K(x, x', t; \Delta)$ as $t\to 0^+$. In particular, given two elliptical operators Δ_1 and Δ_2 of orders $p_1=2v_1$ and $p_2=2v_2$ respectively for some positive integers v_1 and v_2 such that $[\Delta_1, \Delta_2]=0$ with heat kernel satisfying

$$\left(\frac{\partial}{\partial t}+\Delta_1\Delta_2\right)K(x, x', t; \Delta_1\Delta_2)=0 \tag{1.1}$$

and with asymptotic expansion for $t\to 0^+$:

$$K(x, x', t; \Delta_1\Delta_2)\sim\sum_{l=0}^{\infty} a_l(x, x'; \Delta_1\Delta_2)t^{(2l-d)/p} \tag{1.2}$$

then, in d dimensions, the $a_{(1/2)d}$ coefficient of (1.2) satisfies

$$(v_1+v_2)a_{(1/2)d}(x, x; \Delta_1\Delta_2)=\{v_1a_{(1/2)d}(x, x; \Delta_1)+v_2a_{(1/2)d}(x, x; \Delta_2)\} \tag{1.3}$$

for a manifold with or without boundary. When the manifold has a boundary a similar relation holds for the 'boundary' coefficients $c_{(1/2)d}(x, x; \Delta_1\Delta_2)$, $c_{(1/2)d}(x, x; \Delta_1)$ and $c_{(1/2)d}(x, x; \Delta_2)$.

Using this property (1.3) we obtain partial results, in § 3, for the fourth-order operator

$$\Delta=\{\Box^2+E^\kappa\nabla_\kappa\Box+A^{(\kappa\varepsilon)}\nabla_\kappa\nabla_\varepsilon+C^\kappa\nabla_\kappa+Z\} \tag{1.4}$$

and $a_{(1/2)d}(\Delta)$ coefficient (i.e. in $d=4$ dimensions). A special case of this operator arises at the first-loop level in the general fourth-order theory of gravity (Barth and Christensen 1983).

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Operators of order p always appear in the loop expansions of an action S of the same order. (Note: by order here we mean the dimensions of an object. A fourth-order object has dimensions of $1/l^4$.) Theories built out of higher-order invariants are known to have better divergence properties, but probably also various diseases such as negative energy states, and non-unitary S matrix (Pais and Uhlenbeck 1950, Stelle 1977, DeWitt 1965, 1967). However, using power-counting methods, DeWitt (1967) has argued that as far as better divergence behaviour is concerned, nothing is to be gained using actions of order greater than four. Interest in such actions built out of invariants quadratic in the Riemann curvature and its contractions has increased lately for many reasons, especially since Stelle (1977) showed that such theories are formally renormalisable. Due to a lack of tools, work on such theories in curved space with coordinate space methods has been limited. There is hope that when more explicit calculations become available, the various diseases can somehow be done away with. Indeed in light of recent work by Tomboulis (1985), there is reason to believe that in full curved spacetime questions of unitarity must be more carefully answered than otherwise. This paper partially develops, in § 3, a necessary expression for the a_2 coefficient (in four dimensions) of the operator (1.4) which (with E^κ equal to zero) appears at the first-loop level in the general fourth-order theory of gravity (Barth and Christensen 1983). The expression for the integral of the a_2 coefficient is completely solved for. This generalises work by Gilkey (1980) and Christensen (1982) and provides an alternative derivation of these results to those of Barvinsky and Vilkovisky (1985) and Fradkin and Tseytlin (1981, 1982).

Finally, in § 4, we discuss our results and the literature.

2. Useful properties

First we provide some necessary background. Consider some elliptical operator Δ with spectrum $\{\lambda_i, \phi_i\}$ of eigenvalues and eigenfunctions such that the eigenfunctions form a complete orthonormal system and solve the eigenvalue problem $\Delta\phi_i = \lambda_i\phi_i$. Let n be the number of zero modes of the operator and m the number of non-zero modes. Now consider the kernel $K(x, x', t; \Delta)$ of (1.1). It has the formal solution

$$K(x, x', t; \Delta) = \sum_{i=0}^{\infty} \phi_i(x)\phi_i(x')\,\mathrm{e}^{-\lambda_i t}. \tag{2.1}$$

Due to the orthogonality of the eigenfunctions ϕ_i we can write

$$\mathrm{Tr}\int_M \mathrm{d}^d x\, g^{1/2} K(x, x, t; \Delta) = \sum_{i=0}^{\infty} \mathrm{e}^{-\lambda_i t} = \mathrm{Tr}\,\mathrm{e}^{-\Delta t}. \tag{2.2}$$

Now we define $k'(t, \Delta) \equiv \mathrm{Tr}\int_M \mathrm{d}^d x\, g^{1/2} K(x, x, t; \Delta)$ where the prime denotes the exclusion of the n zero modes. Then as $t \to 0^+$ we have

$$\begin{aligned} k'(t, \Delta) &= \mathrm{Tr}\,\mathrm{e}^{-\Delta t} = \sum_{i=n+1}^{\infty} \mathrm{e}^{-\lambda_i t} \\ &\sim \sum_{l=0}^{\infty} \int_M \mathrm{d}^d x\, g^{1/2} a'_l(x, x, \Delta)\, t^{(2l-d)/p} \\ &\equiv \sum_{l=0}^{\infty} A'_l(\Delta)\, t^{(2l-d)/p} \end{aligned} \tag{2.3}$$

which defines the coefficients $\Delta'_l(\Delta)$.

Recalling the integral representation for the gamma function

$$\Gamma(s)=\int_0^\infty t^{s-1}\,\mathrm{e}^{-t}\,\mathrm{d}t \tag{2.4}$$

and using the Mellin transform of (2.3) we obtain

$$\int_0^\infty \mathrm{d}t\, t^{s-1}k'(t,\Delta)=\sum_{i=n+1}^{\infty}\lambda_i^{-s}\Gamma(s). \tag{2.5}$$

Defining the zeta function

$$\zeta(s,\Delta)=\sum_{i=n+1}^{\infty}\lambda_i^{-s} \tag{2.6}$$

then from (2.5) we have

$$\zeta(s,\Delta)=\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\, t^{s-1}k'(t,\Delta). \tag{2.7}$$

Equation (2.7) allows us to relate the zeta function to the asymptotic expansion coefficients in (1.2) and (2.3). Using suitable analytic continuation (Dowker and Critchley 1976, Hawking 1977) it is easily shown that

$$\zeta(0,\Delta)=m=A'_{(1/2)d}(\Delta) \tag{2.8}$$

where m is the number of non-zero modes of the operator Δ as mentioned above. From (2.6) we have immediately, for some arbitrary positive number k,

$$\zeta(s,\Delta^k)=\zeta(ks,\Delta) \tag{2.9}$$

and therefore together with (2.8)

$$A_{(1/2)d}(\Delta^k)=A_{(1/2)d}(\Delta) \tag{2.10}$$

where $A_l(\Delta)=\{A'_l(\Delta)+n\}$, n being the number of zero modes of the operator Δ. Later we show that (2.10) holds for the $a_{(1/2)d}$ coefficients as well.

Consider now the determinant of the operator Δ. Formally (although this can be made mathematically correct)

$$\det\Delta=\prod_{i=n+1}^{\infty}\lambda_i \tag{2.11}$$

which, together with the definition of the zeta function (2.6), gives

$$\ln\det\Delta=-\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\zeta(s,\Delta). \tag{2.12}$$

Using (2.7) and the fact that

$$\lim_{s\to 0}1/\Gamma(s)=(s+\gamma s^2+\ldots) \tag{2.13}$$

we obtain

$$\ln\det\Delta=-\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\left((s+\gamma s^2+\ldots)\int_0^\infty \mathrm{d}t\, t^{s-1}\,k'(t,\Delta)\right). \tag{2.14}$$

This expression (2.14) has divergent parts. Using (2.3) it is not hard to show that the coefficients of the parts diverging as $t^{(2l-d)/p}$ are

$$-\frac{p}{(2l-d)}A_l'(\Delta) \qquad \text{for } (2l-d)<0 \tag{2.15a}$$

and for the part diverging as ln t the coefficient is

$$-A'_{(1/2)d}(\Delta). \tag{2.15b}$$

This is already well known for $d=4$ dimensions (Hawking 1977). Expression (2.15) however is not quite correct and must be slightly modified if it is to be consistent with (2.10). This becomes clear when we realise that

$$\ln \det \Delta^k = k \ln \det \Delta. \tag{2.16}$$

Then from dimensional considerations the logarithmic divergences in (2.16) must be related among themselves as

$$A'_{(1/2)d}(\Delta^k) = kA'_{(1/2)d}(\Delta). \tag{2.17}$$

This is not consistent with (2.10). The resolution of the difficulty follows if we define a new asymptotic expansion coefficient proportional to the old one. That is, we define

$$\tilde{A}_l'(\Delta) \equiv (2/p)A_l'(\Delta) \tag{2.18}$$

where p is the order of the operator Δ. This can always be done because $\frac{1}{2}p$ is just a number which does not affect the asymptotic expansion in any way. (This may, of course affect the counterterms however; we return to this in § 4.) When this is done then (2.17) becomes

$$\tfrac{1}{2}pk\tilde{A}'_{(1/2)d}(\Delta^k) = k[\tfrac{1}{2}p\tilde{A}'_{(1/2)d}(\Delta)] \tag{2.19}$$

which is consistent with (2.10).

We are now able to derive our main result. Consider the elliptical operators Δ_1 and Δ_2 of orders $p_1=2v_1$, $p_2=2v_2$ respectively such that $[\Delta_1, \Delta_2]=0$ and $\Delta_0=\Delta_1\Delta_2$ with spectra $\{\Delta_0\colon \lambda_0=\lambda_1\lambda_2, \phi\}$, $\{\Delta_1\colon \lambda_1, \phi\}$, $\{\Delta_2, \lambda_2, \phi\}$. Then due to the fact that

$$\ln \det \Delta_0 = \ln \det \Delta_1\Delta_2 = \{\ln \det \Delta_1 + \ln \det \Delta_2\} \tag{2.20}$$

comparison of the logarithmic divergences, together with (2.19) and including the zero modes such that $\tilde{A}_l=\{\tilde{A}_l'+n\}$, demands that

$$(v_1+v_2)\tilde{A}_{(1/2)d}(\Delta_1\Delta_2) = \{v_1\tilde{A}_{(1/2)d}(\Delta_1) + v_2\tilde{A}_{(1/2)d}(\Delta_2)\}. \tag{2.21}$$

This result holds true even when the manifold has boundaries provided the asymptotic expansion (1.2) is supplemented by 'boundary' coefficients c_l (Gilkey 1980) such that (we now drop the tilde over the $\tilde{A}_l$ coefficients)

$$A_l(\Delta) = \left(\int_M d^dx\, g^{1/2}a_l(x,x,\Delta) + \int_{\partial M} d^{d-1}x\, \gamma^{1/2}c_l(x,x,\Delta)\right) \tag{2.22}$$

where ∂M is the boundary of M, and $\gamma^{1/2}\,d^{d-1}x$ is the invariant surface element with $\gamma_{\alpha\beta}$ the induced surface metric. (For details see York (1979) and references therein. For recent work with boundaries and higher-order theories of gravity see Barth (1985).) From (2.22) it is clear that the c_l coefficients are of one order less than the a_l coefficients. The $a_{(1/2)d}$ coefficient is always of even order for manifolds of even dimension, and odd for manifolds of odd dimension. When d is odd, then the $A_{(1/2)d}(\Delta)$ coefficient is completely determined by the boundary coefficient $c_{(1/2)d}$ term. (Thus for compact manifolds without boundary and of odd dimension the $A_{(1/2)d}(\Delta)$ coefficient is always zero.) This shows that even when there are boundaries, it is not possible to construct volume invariants of odd order. Thus on dimensional grounds, when $d=$ even dimensions, the $a_{(1/2)d}$ and $c_{(1/2)d}$ coefficients implicit in (2.22) must be independent of one another. That is using (2.21) and (2.22) we have

$$(v_1+v_2)a_{(1/2)d}(x,x;\Delta_1\Delta_2)=\{v_1 a_{(1/2)d}(x,x;\Delta_1)+v_2 a_{(1/2)d}(x,x;\Delta_2)\} \tag{2.23a}$$

$$(v_1+v_2)c_{(1/2)d}(x,x;\Delta_1\Delta_2)=\{v_1 c_{(1/2)d}(x,x;\Delta_1)+v_2 c_{(1/2)d}(x,x;\Delta_2)\}. \tag{2.23b}$$

Notice that on dimensional grounds, the extrinsic curvature $K_{\alpha\beta}$ must be used to build every surface invariant in the $c_{(1/2)d}$ coefficient. Thus setting $K_{\alpha\beta}$ equal to zero everywhere (i.e. working with a flat boundary) makes all the surface contributions to the $A_{(1/2)d}$ coefficient vanish. This is all the more clear when working with operators whose leading symbols are some power of the metric tensor. The general operator of this form is

$$\Delta=\{(\Box)^k+N^{(\alpha_1\ldots\alpha_{2k-1})}\nabla_{\alpha_1}\nabla_{\alpha_2}\ldots\nabla_{\alpha_{2k-1}}+\ldots+N^{\alpha_1}\nabla_{\alpha_1}+N\}. \tag{2.24}$$

For such operators all available tensors from which the invariants of the $a_{(1/2)d}$ coefficients can be built (i.e. the Riemann curvature tensor, its contractions, the tensors $N^{(\alpha\ldots\beta)}$ in (2.24) and the use of covariant derivation) of even rank are of even order, and all tensors of odd rank are of odd order with one exception. The extrinsic curvature tensor $K_{\alpha\beta}$ is the only tensor of even rank but odd order (it is a first-order object). Thus all of the surface invariants in the $c_{(1/2)d}$ coefficient must have at least one extrinsic curvature tensor, or its contraction. This is because the surface invariants (for $l=$ even number) are of odd order (in fact they are of order $(2l-1)$ (see below)). Due to this (2.23a) holds whether the manifold has a boundary or not.

We would also like to draw attention to the fact that unless the operators Δ_1 and Δ_2 commute, it is not in general possible to relate their eigenvalues and eigenfunctions to the eigenvalues and eigenfunctions of the operator formed from their product. That is, for non-commuting elliptical operators Δ_1 and Δ_2 one, in general, does not have the relations

$$\Delta=\Delta_1\Delta_2 \qquad \lambda=\lambda_1\lambda_2 \qquad \phi=\phi_1=\phi_2. \tag{2.25}$$

Finally in this section, we mention the following. Using simple arguments it is possible to break up any operator of the form (2.24) into a trivial and non-trivial part depending on which asymptotic coefficient is being calculated. First we observe from Gilkey's (1980) lemma 2.2 and from (2.23) that the order of the invariants making up an asymptotic coefficient is independent of the order of the operator. Therefore using results for second-order operators of the form (2.24) we immediately have (DeWitt 1965, Gilkey 1975)

$$\mathrm{O}(a_l)=2l \tag{2.26}$$

(we use the notation $\mathrm{O}(a_l)$ to means 'the order of (a_l)'). From Gilkey's lemma it also follows that (2.26) is independent of the dimension of the manifold. Therefore when calculating the expansion coefficient a_l for an operator of the form (2.24), the terms in (2.24) with tensors $N^{(\alpha\ldots\beta)}$ of order greater than $2l$ are uninteresting because they can never be used to build up invariants in the asymptotic coefficient a_l. More succinctly one has the immediate relationship

$$[a_l((\Box)^k + N^{(\alpha_1 \ldots \alpha_{2k-1})}\nabla_{\alpha_1}\ldots\nabla_{\alpha_{2k-1}} + \ldots + N^{\alpha_1}\nabla_{\alpha_1} + N)]$$
$$= [a_l((\Box)^k + N^{(\alpha_1 \ldots \alpha_{2k-1})}\nabla_{\alpha_1}\ldots\nabla_{\alpha_{2k-1}} + \ldots + N^{(\alpha_1 \ldots \alpha_{2k-2l})}\nabla_{\alpha_1}\ldots\nabla_{\alpha_{2k-2l}})]. \quad (2.27)$$

This holds true regardless of the dimension of the manifold.

3. A fourth-order exampole

We now use (2.23) to obtain some results for $a_2(\Delta)$ where Δ is the fourth-order operator

$$\Delta = \{\Box^2 + E^\kappa\nabla_\kappa\Box + A^{(\kappa\varepsilon)}\nabla_\kappa\nabla_\varepsilon + C^\kappa\nabla_\kappa + Z\} \quad (3.1)$$

and the objects E^κ, $A^{(\kappa\varepsilon)}$, C^κ and Z all commute with each other and with $Y_{\kappa\varepsilon}$ (see notation section in Barth 1987 (referred to as I)). Then from (3.1) and relation (2.26), together with the commutation relations of the tensors E^κ, $A^{(\kappa\varepsilon)}$, C^κ, Z and $Y_{\kappa\epsilon}$ just mentioned, the general form for $a_2(\Delta)$ must be (see also Christensen (1982), Gilkey (1980))

$$\begin{aligned}180[a_2(\Delta)] = \{&r_1 R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + r_2 R_{\alpha\beta}R^{\alpha\beta} + r_3 R^2 + r_4 R_{;\kappa}{}^{\kappa} + r_5 Y_{\alpha\beta}Y^{\alpha\beta} + e_1 RE^\kappa{}_{;\kappa}\\
&+ e_2 RE^\kappa E_\kappa + e_3 R_{;\kappa}E^\kappa + e_4 E^\kappa{}_;{}^\varepsilon R_{\kappa\varepsilon} + e_5 E^\kappa E^\varepsilon R_{\kappa\varepsilon} + e_6 E^\kappa{}_{;\kappa}{}^\varepsilon{}_\varepsilon\\
&+ e_7 E^\kappa{}_;{}^\varepsilon E_{\kappa;\varepsilon} + e_8 E^\kappa{}_;{}^\varepsilon E_{\varepsilon;\kappa} + e_9 E^\kappa{}_{;\kappa}E^\varepsilon{}_{;\varepsilon} + e_{10}E^\kappa E^\varepsilon{}_{;\varepsilon\kappa} + e_{11}E^\kappa E_{k;}{}^\varepsilon{}_\varepsilon\\
&+ e_{12}E^\kappa E_\kappa E^\varepsilon{}_{;\varepsilon} + e_{13}E_{\kappa;\varepsilon}E^\kappa E^\varepsilon + e_{14}E^\kappa E_\kappa E^\varepsilon E_\varepsilon + e_{15}E^\kappa E^\varepsilon A_{\kappa\varepsilon} + e_{16}E^\kappa{}_;{}^\varepsilon A_{\kappa\varepsilon}\\
&+ e_{17}E^\kappa A_{\kappa\varepsilon;}{}^\varepsilon + e_{18}E^\kappa E_\kappa A + e_{19}E^\kappa{}_{;\kappa}A + e_{20}E^\kappa A_{;\kappa} + e_{21}E^\kappa C_\kappa\\
&+ e_{22}E^\kappa{}_;{}^\varepsilon Y_{\kappa\varepsilon} + e_{23}E^\kappa Y_{\kappa\varepsilon;}{}^\varepsilon + a_1' A^{\kappa\varepsilon}{}_{;\kappa\varepsilon} + a_2' A_\kappa{}^\kappa + a_3' A^{\kappa\varepsilon}R_{\kappa\varepsilon} + a_4' A^{\kappa\varepsilon}A_{\kappa\varepsilon}\\
&+ a_5' AR + a_6' A^2 + c_1 C^\kappa{}_{;\kappa} + \zeta_1 Z\}\end{aligned} \quad (3.2)$$

where $g^{\kappa\varepsilon}A_{\kappa\varepsilon} = A$, and the real coefficients $e_1, \ldots, e_{23}$, $a_1', \ldots, a_6'$, c_1 and z_1 are to be determined. The primes above the coefficients a_1' etc are there to distinguish them from the $a_2(\Delta)$ asymptotic expansion coefficient. Then using the expression from I for $a_2(-\Box + B^\kappa\nabla_\kappa + X)$ (namely (4.12)) and the relationship (2.23) we can determine many of the coefficients in (3.1). We do this by considering the following three cases.

Case 1: $\Delta = \Delta_1\Delta_2 = (-\Box + B^\kappa\nabla_\kappa + X)(-\Box + B^\varepsilon\nabla_\varepsilon + X)$. The resulting operator is of the form (3.1) with

$$E^\kappa = -2B^\kappa$$
$$A^{(\kappa\varepsilon)} = (B^\kappa B^\varepsilon - 2Xg^{\kappa\varepsilon} - B^\kappa{}_;{}^\varepsilon - B^\varepsilon{}_;{}^\kappa)$$
$$C^\kappa = (B_\varepsilon B^\kappa{}_;{}^\varepsilon + 2B^\kappa X - B^\kappa{}_{;\varepsilon}{}^\varepsilon - B_\varepsilon R^{\kappa\varepsilon} - 2X_;{}^\kappa)$$
$$Z = (X^2 + B^\kappa X_{;\kappa} - X_{;\kappa}{}^\kappa).$$

Case 2: $\Delta = \Delta_1\Delta_2 = (-\Box + B^\kappa\nabla_\kappa + X)(-\Box + Q^\varepsilon\nabla_\varepsilon + Y)$. The resulting operator is of the form (3.1) with

$$E^\kappa = -(B^\kappa + Q^\kappa)$$
$$A^{(\kappa\varepsilon)} = (\tfrac{1}{2}B^\kappa Q^\varepsilon + \tfrac{1}{2}B^\varepsilon Q^\kappa - Xg^{\kappa\varepsilon} - Yg^{\kappa\varepsilon})$$
$$C^\kappa = (B^\kappa Y + Q^\kappa X)$$
$$Z = XY$$

where, in order to ensure commutivity of the operators Δ_1 and Δ_2 we have required $R_{\alpha\beta\gamma\delta}=0=B_{\kappa;\varepsilon}=Q_{\kappa;\varepsilon}=X_{;\kappa}=Y_{;\kappa}$.

Case 3: $\Delta=\Delta_1\Delta_2=(-\Box+B^{\kappa}\nabla_{\kappa})(-\Box+Q^{\varepsilon}\nabla_{\varepsilon})$. The resulting operator is of the form (3.1) with

$$E^{\kappa}=-(B^{\kappa}+Q^{\kappa})$$
$$A^{(\kappa\varepsilon)}=(\tfrac{1}{2}B^{\kappa}Q^{\varepsilon}+\tfrac{1}{2}B^{\varepsilon}Q^{\kappa}-B^{\kappa}{}_{;}{}^{\varepsilon}-B^{\varepsilon}{}_{;}{}^{\kappa})$$
$$C^{\kappa}=(B_{\varepsilon}Q^{\kappa}{}_{;}{}^{\varepsilon}-Q^{\kappa}{}_{;\varepsilon}{}^{\varepsilon}-Q_{\varepsilon}R^{\kappa\varepsilon})$$

where, in order to ensure commutivity of the operators Δ_1 and Δ_2 we have required

$$B_{(\kappa;\varepsilon)}=Q_{(\kappa;\varepsilon)}$$
$$(B_{\kappa}R^{\kappa\varepsilon}-Q_{\kappa}R^{\kappa\varepsilon}+B^{\varepsilon}{}_{;\kappa}{}^{\kappa}-Q^{\varepsilon}{}_{;\kappa}{}^{\kappa}+B^{\kappa}Q^{\varepsilon}{}_{;\kappa}-Q^{\kappa}B^{\varepsilon}{}_{;\kappa})=0.$$

These three cases satisfy the requirement that $[\Delta_1,\Delta_2]=0$. Thus comparing invariants in the resulting expression (3.2) with (4.12) in I gives straightforwardly the following relations:

case 1:

1. $r_1=1$
2. $r_2=-1$
3. $r_3=\frac{5}{2}$
4. $r_4=6$
5. $r_5=15$
6. $(-2e_1-2a'_5)=15$
7. $(4e_2+a'_5)=-\frac{15}{2}$
8. $(-2e_3-a'_1-c_1)=0$
9. $(-2e_4-2a'_1-2a'_3-2c_1)=0$
10. $(4e_5+2e_{17}+2e_{21}+a'_1+a'_3+c_1)=0$
11. $(-2e_6-2a'_1-2a'_2-c_1)=15$
12. $(4e_7+2e_{16}+2a'_2+2a'_4)=-\frac{15}{2}$
13. $(4e_8+2e_{16}+a'_1+2a'_4+c_1)=-\frac{15}{2}$
14. $(4e_9+4e_{19}+a'_1+4a'_6)=\frac{45}{2}$
15. $(4e_{10}+2e_{17}+4e_{20}+2a'_1+c_1)=0$
16. $(4e_{11}+2e_{17}+2e_{21}+2a'_2)=-15$
17. $(-8e_{12}-2e_{17}-8e_{18}-2e_{19}-4a'_6)=-\frac{45}{2}$
18. $(-8e_{13}-8e_{15}-2e_{16}-2e_{17}-4e_{20}-2e_{21}-4a'_4)=0$
19. $(16e_{14}+4e_{15}+4e_{18}+a'_4+a'_6)=\frac{45}{8}$
20. $(-8e_{15}-32e_{18}-4e_{21}-4a'_4-16a'_6)=45$
21. $(4e_{16}+16e_{19}+8a'_4+32a'_6+2c_1)=-90$
22. $(4e_{17}+16e_{20}+4e_{21}+2c_1+z_1)=0$
23. $(-2a'_3-8a'_5)=-30$
24. $(16a'_4+64a'_6+z_1)=90$
25. $(-2a'_1-8a'_2-2c_1-z_1)=-30$
26. $e_{22}=-\frac{45}{2}$
27. $e_{23}=-\frac{15}{4}$;

case 2:

28. $e_2=-\frac{15}{4}$
29. $e_{14}=\frac{45}{16}$
30. $(4e_{14}+e_{15}+e_{18})=0$

31. $(e_{15}+4e_{18})=-\frac{45}{2}$
32. $(e_{15}+4e_{18}+e_{21})=0$;

case 3:

33. $(e_7-e_8)=0$
34. $(2e_{13}+2e_{15}+e_{17}+e_{21})=0$
35. $(2e_{12}+1/2e_{17}+2e_{18})=\frac{45}{4}$.

These equations form 35 independent equations in 36 variables. It is possible to solve for many of them. When we do this we obtain the results:

$$\begin{aligned}180[a_2(\Delta)]=\{&|R_{\alpha\beta\gamma\delta}|^2-|R_{\alpha\beta}|^2+\tfrac{5}{2}R^2+6R_{;\kappa}{}^{\kappa}+15Y_{\kappa\varepsilon}Y^{\kappa\varepsilon}-15RE^{\kappa}{}_{;\kappa}-\tfrac{15}{4}RE^{\kappa}E_{\kappa}\\&-10E^{\kappa}R_{;\kappa}-5E^{\kappa}{}_{;}{}^{\varepsilon}R_{\kappa\varepsilon}-\tfrac{1}{4}(50+c_1)E^{\kappa}E^{\varepsilon}R_{\kappa\varepsilon}+(-75+c_1)E^{\kappa}{}_{;\kappa}{}^{\varepsilon}{}_{\varepsilon}\\&+\tfrac{1}{4}(-65+c_1)E^{\kappa}{}_{;}{}^{\varepsilon}E_{\kappa;\varepsilon}+\tfrac{1}{4}(-65+c_1)E^{\kappa}{}_{;}{}^{\varepsilon}E_{\varepsilon;\kappa}+\tfrac{1}{4}(55+c_1)E^{\kappa}{}_{;\kappa}E^{\varepsilon}{}_{;\varepsilon}\\&+\tfrac{1}{4}(-40+c_1)E^{\kappa}E_{\varepsilon;}{}^{\varepsilon}{}_{\kappa}+\tfrac{1}{4}(-80+c_1)E^{\kappa}E_{\kappa;}{}^{\varepsilon}{}_{\varepsilon}+\tfrac{1}{8}(75+c_1)E^{\kappa}E_{\kappa}E^{\varepsilon}{}_{;\varepsilon}\\&+\tfrac{1}{4}(-15+c_1)E^{\kappa}E^{\varepsilon}E_{\kappa;\varepsilon}+\tfrac{45}{16}(E^{\kappa}E_{\kappa})^2-\tfrac{15}{2}E^{\kappa}E^{\varepsilon}A_{\kappa\varepsilon}+\tfrac{1}{2}(30-c_1)E^{\kappa}{}_{;}{}^{\varepsilon}A_{\kappa\varepsilon}\\&-\tfrac{1}{2}c_1E^{\kappa}A_{\kappa\varepsilon;}{}^{\varepsilon}-\tfrac{15}{4}E^{\kappa}E_{\kappa}A-15E^{\kappa}{}_{;\kappa}A+\tfrac{45}{2}E^{\kappa}C_{\kappa}-\tfrac{45}{2}E^{\kappa}{}_{;}{}^{\varepsilon}Y_{\kappa\varepsilon}\\&-\tfrac{15}{4}E^{\kappa}Y_{\kappa\varepsilon,}{}^{\varepsilon}+(20-c_1)A_{\kappa\varepsilon;}{}^{\kappa\varepsilon}+10A_{;\kappa}{}^{\kappa}-15A^{\kappa\varepsilon}R_{\kappa\varepsilon}+\tfrac{15}{4}A_{\kappa\varepsilon}A^{\kappa\varepsilon}\\&+\tfrac{15}{2}AR+\tfrac{15}{8}A^2+c_1C^{\kappa}{}_{;\kappa}-90Z\}.\end{aligned}\tag{3.3}$$

We draw attention to the fact that the coefficient e_{20} equals zero and therefore the invariant $E^{\kappa}A_{;\kappa}$ does not appear in (3.3). Equation (3.3), although as it stands is not completely solved, in fact gives enough information to solve for the integral of (3.2). When we do this we obtain

$$\begin{aligned}180[A_2(\Delta)]=\int_M g^{1/2}\,\mathrm{d}^4x\{&|R_{\alpha\beta\gamma\delta}|^2-|R_{\alpha\beta}|^2+\tfrac{5}{2}R^2+15Y_{\alpha\beta}Y^{\alpha\beta}-\tfrac{15}{2}RE^{\kappa}{}_{;\kappa}-\tfrac{15}{4}RE^{\kappa}E_{\kappa}\\&+3E^{\kappa}E^{\varepsilon}R_{\kappa\varepsilon}+\tfrac{15}{2}E^{\kappa}{}_{;\kappa}E^{\varepsilon}{}_{;\varepsilon}-\tfrac{15}{4}E^{\kappa}E_{\kappa;}{}^{\varepsilon}{}_{\varepsilon}-\tfrac{45}{2}E^{\kappa}E^{\varepsilon}E_{\kappa;\varepsilon}+\tfrac{45}{16}(E^{\kappa}E_{\kappa})^2\\&-\tfrac{15}{2}E^{\kappa}E^{\varepsilon}A_{\kappa\varepsilon}+15E^{\kappa}{}_{;}{}^{\varepsilon}A_{\kappa\varepsilon}-\tfrac{15}{4}E^{\kappa}E_{\kappa}A-15E^{\kappa}{}_{;\kappa}A+\tfrac{45}{2}E^{\kappa}C_{\kappa}\\&-\tfrac{75}{4}E^{\kappa}{}_{;}{}^{\varepsilon}Y_{\kappa\varepsilon}-15A^{\kappa\varepsilon}R_{\kappa\varepsilon}+\tfrac{15}{4}A^{\kappa\varepsilon}A_{\kappa\varepsilon}+\tfrac{15}{2}AR+\tfrac{15}{8}A^2-90Z\}.\end{aligned}\tag{3.4}$$

Note that group indices are implicit in (3.2)-(3.4). The case of (3.4) with E^{κ} equal to zero was first obtained by Gilkey (1980). He used methods different from ours here and although they are useful, they do not lend themselves to finding (3.4) as easily as does relation (2.23).

It is clear that if one more independent equation could be found then (3.3) would be completely solved. An obvious question is whether any of the clever methods of Gilkey's could be used. Unfortunately they cannot because they are associated with the integral of the a_2 coefficient, something already determined in (3.4). One might also suggest an additional 'case 4' to those considered above where

$$\Delta=\Delta_1\Delta_2=(-\Box+B^{\kappa}\nabla_{\kappa}+X)(-\Box+Q^{\varepsilon}\nabla_{\varepsilon}+Y)$$

so that (3.1) takes the form

$$\begin{aligned}&E^{\kappa}=-(B^{\kappa}+Q^{\kappa})\\&A^{(\kappa\varepsilon)}=(\tfrac{1}{2}B^{\kappa}Q^{\varepsilon}+\tfrac{1}{2}B^{\varepsilon}Q^{\kappa}-B^{\kappa}{}_{;}{}^{\varepsilon}-B^{\varepsilon}{}_{;}{}^{\kappa}-Xg^{\kappa\varepsilon}-Yg^{\kappa\varepsilon})\\&C^{\kappa}=(B^{\varepsilon}Q^{\kappa}{}_{;\varepsilon}-Q^{\kappa}{}_{;\varepsilon}{}^{\varepsilon}-2Y_{;}{}^{\kappa}+B^{\kappa}Y+Q^{\kappa}X)\\&Z=(B^{\kappa}Y_{,\kappa}+XY-Y_{,\kappa}{}^{\kappa}).\end{aligned}$$

In addition one must require that Δ_1 and Δ_2 commute and therefore demand the additional relations:

$$B_{(\kappa;\varepsilon)} = Q_{(\kappa;\varepsilon)}$$
$$(B^{\kappa}{}_{;\varepsilon}{}^{\varepsilon} - Q^{\kappa}{}_{;\varepsilon}{}^{\varepsilon} + B^{\varepsilon}Q^{\kappa}{}_{;\varepsilon} - Q^{\varepsilon}B^{\kappa}{}_{;\varepsilon} + 2X_{;}{}^{\kappa} - 2Y_{;}{}^{\kappa}) = 0$$
$$(B^{\kappa}Y_{;\kappa} - Q^{\kappa}X_{;\kappa} + X_{;\kappa}{}^{\kappa} - Y_{;\kappa}{}^{\kappa}) = 0.$$

This 'case 4' is the most general case possible. However, it turns out that there are no tensors B^{κ}, Q^{κ}, X and Y such that these restrictions are satisfied and also yield a new independent equation. This is easily shown by the fact that any 'new' equation is always one incompatible with those already obtained. Put another way, the restrictions are so strong that the tensors B^{κ}, Q^{κ}, X, Y and their derivatives are related such that no new invariant combinations not already found in the cases 1–3 occur. For this reason case 4 is uninteresting.

Most other methods used in calculating these asymptotic expansion coefficients deal only with the integral of the coefficient and so are not useful to us here. There is no ansatz or recursive method like the Schwinger–DeWitt ansatz so useful in the case of second-order operators (DeWitt 1965). The method of 'doubling' the manifold (Gilkey 1974, McKean and Signer 1967), while extremely useful for the case of second-order operators, is problematic in its application to higher-order operators. If we define a new manifold which is the product of two other manifolds such that the metric is

$$g_M = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \tag{3.5}$$

then

$$g_M^{\alpha\beta}\nabla_\alpha\nabla_\beta \equiv \underset{M}{\Box} = (g_1^{ab}\nabla_a\nabla_b + g_2^{xy}\nabla_x\nabla_y) = \left(\underset{1}{\Box} + \underset{2}{\Box}\right).$$

That is, the operator on the full manifold splits into two separate operators each defined on one of the two submanifolds. It is then possible to relate the asymptotic expansion coefficients for the operators on the submanifolds with those on the full manifold. (Note that the operators all commute with each other.) However with higher-order operators things are not so convenient. Consider for example the fourth-order case. Then we have

$$\underset{M}{\Box}^2 = \left(\underset{1}{\Box} + \underset{2}{\Box}\right)^2 = \left(\underset{1}{\Box}^2 + 2\underset{1}{\Box}\,\underset{2}{\Box} + \underset{2}{\Box}^2\right). \tag{3.6}$$

The cross terms do not allow the operator $\underset{M}{\Box}^2$ to split neatly as in the second-order case, hence this method does not give us any new information.

One might also attempt to obtain new information by considering

$$[a_2(\Box^4 + \theta^{\kappa}\nabla_\kappa\Box^3 + M^{(\kappa\varepsilon)}\nabla_\kappa\nabla_\varepsilon\Box^2 + N^{(\alpha\beta\kappa)}\nabla_\kappa\nabla_\beta\nabla_\kappa\Box + K^{(\alpha\beta\kappa\varepsilon)}\nabla_\alpha\nabla_\beta\nabla_\kappa\nabla_\varepsilon)]$$

(from (2.27) we note that the other terms in this eighth-order operator are uninteresting) or still other higher-order operators. Unfortunately this also yields nothing. Indeed one can show that all of these sorts of higher-order cases with operators of the form

$$\{(-\Box)^k + \theta^{\kappa}\nabla_\kappa\Box^{k-1} + M^{(\kappa\varepsilon)}\nabla_\kappa\nabla_\varepsilon\Box^{k-2} + N^{(\alpha\beta\kappa)}\nabla_\alpha\nabla_\beta\nabla_\kappa\Box^{k-3}$$
$$+ K^{(\alpha\beta\kappa\varepsilon)}\nabla_\alpha\nabla_\beta\nabla_\kappa\nabla_\varepsilon\Box^{k-4} + \ldots\} \tag{3.7}$$

(these are the type produced from operator products such as $(-\Box+B^{\kappa}\nabla_{\kappa}+X)^{k}$) are determined by the fourth-order case given in expression (3.2). It is appropriate to mention here the work of Gilkey (1980) where, for a specific form of operator of arbitrary order and on manifolds of arbitrary dimension, he obtains an expression for the integral of the a_2 coefficient. In our paper here we have chosen not to add these additional 'degrees of freedom' to the calculation, although inclusion of arbitrary dimensions and order operators can be accomplished following Gilkey's example.

4. Discussion

In the general fourth-order theory of gravity, with action given by

$$S=\int_M \mathrm{d}^4x\, g^{1/2}\{\rho(R-2\Lambda)+\alpha R^2+\beta R_{\mu\nu}R^{\mu\nu}\}+\varepsilon\chi \tag{4.1}$$

the operator

$$\Delta=(\Box^2+A^{(\kappa\varepsilon)}\nabla_{\kappa}\nabla_{\varepsilon}+C^{\kappa}\nabla_{\kappa}+Z) \tag{4.2}$$

appears, in addition to two second-order operators, at the first-loop level (Barth and Christensen 1983). Operator (4.2) is a special case of (3.1) and provides at least one reason why (3.1) is of interest. We also note that although we assumed $[E^{\kappa}, E^{\varepsilon}]=0$, and $[E^{\kappa}, A^{(\alpha\beta)}]=0$, etc, in equations (3.1)-(3.4), when we specialise to the case (4.2) we obtain the general expression for the a_2 coefficient of (4.2), even when it is built up of non-commuting tensors $A^{(\alpha\beta)}$, C^{κ} and Z.

There has been other work on the a_2 coefficient for higher-order operators (Barvinsky and Vilkovisky 1985, Christensen 1982, Fradkin and Tseytlin 1981, 1982, 1985, Gilkey 1980). As we have mentioned, Gilkey (1980) using means other than those of (2.23) obtained $A_2(\Box^2+A^{(\kappa\varepsilon)}\nabla_{\kappa}\nabla_{\varepsilon}+C^{\kappa}\nabla_{\kappa}+Z)$. Deleting terms with E^{κ} everywhere in our expression (3.4) rederives Gilkey's result. However using (2.23) it has been possible to extend these results to the more general case $A_2(\Box^2+E^{\kappa}\nabla_{\kappa}\Box+A^{(\kappa\varepsilon)}\nabla_{\kappa}\nabla_{\varepsilon}+C^{\kappa}\nabla_{\kappa}+Z)$ and, to within one unknown $a_2(\Box^2+E^{\kappa}\nabla_{\kappa}\Box+A^{(\kappa\varepsilon)}\nabla_{\kappa}\nabla_{\varepsilon}+C^{\kappa}\nabla_{\kappa}+Z)$. Furthermore we obtained results for the total divergences in $a_2(\Box^2+E^{\kappa}\nabla_{\kappa}\Box+A^{(\kappa\varepsilon)}\nabla_{\kappa}\nabla_{\varepsilon}+C^{\kappa}\nabla_{\kappa}+Z)$ which Gilkey does not.

In an aside, we note that it is well known that, when the manifold is compact without boundary, total divergences lose their importance—at least as far as loop corrections are concerned. Naturally when there are boundaries, this is no longer true. In addition, the connection between the a_2 coefficient and anomalies, which can induce total divergences into the action, are at least two reasons why they can be of interest and why methods to calculate them are of importance. In this regard we mention that the work of Barvinsky and Vilkovisky (1985), as it stands, does not allow one to find total divergences in the a_2 coefficient. As far as we know the method of (2.23) seems to be the only way of doing so.

Fradkin and Tseytlin (1981) claim to have completely solved for the a_2 coefficient for the operator (4.2) including the invariants which can be written as total divergences. Disregarding the total divergences $A^{(\kappa\varepsilon)}{}_{;\kappa\varepsilon}$, $A_{;\kappa}{}^{\kappa}$ and $C^{\kappa}{}_{;\kappa}$ for now, we note that their expressions are all off by a factor of $\frac{1}{2}$. Without this factor their results are not consistent with (2.10). Considering now the total divergnce terms we obtain, after putting these in a form allowing immediate comparison with (3.3) from Fradkin and Tseytlin (1981)

$$180[a_2(\Delta)]=\{\ldots-30A^{\kappa\varepsilon}{}_{;\kappa\varepsilon}+15A_{;\kappa}{}^{\kappa}+180C^{\kappa}{}_{;\kappa}\} \tag{4.3a}$$

and from Fradkin and Tseytlin (1982)

$$180[a_2(\Delta)] = \{\ldots + 60A^{\kappa\varepsilon}{}_{;\kappa\varepsilon} + 15A_{;\kappa}{}^{\kappa}\}. \tag{4.3b}$$

Comparing (4.3) with (3.3) we see that there is no agreement. We attribute this disagreement to the fact that Fradkin and Tseytlin seem not to require that Δ_1 and Δ_2 commute with each other as they must for an expression like (2.23) to be true. In this connection we mention that although we have built upon the papers by Fradkin and Tseytlin (1981, 1982), expression (2.23) is not the same as a similar one obtained there.

Disregarding the factor of two and the total divergences, Fradkin and Tseytlin obtain their expressions for the a_2 coefficient of (4.2) using only second-order operators of the general form $(-\Box + X)$ (instead of $(-\Box + B^\kappa \nabla_\kappa + X)$ as we did in cases 1-3), as well as the method of doubling the manifold. We have already said in § 3 that doubling the manifold does not seem useful in connection with higher-order operators. As far as using (2.23) and only second-order operators of the form $(-\Box + X)$ is concerned, when this is done the fourth-order operator obtained is always of the form (4.2). Thus deleting the terms e_1, e_2, through e_{23} in (3.2) we obtain the general form for $a_2(\Box^2 + A^{(\kappa\varepsilon)}\nabla_\kappa\nabla_\varepsilon + C^\kappa\nabla_\kappa + Z)$. In a similar way, many of the 35 equations in 36 unknowns must also be deleted. In fact all that is left of these equations is the subset:

1. $r_1 = 1$
2. $r_2 = -1$
3. $r_3 = \frac{5}{2}$
4. $r_4 = 6$
5. $r_5 = 15$
6. $(-2a_3' - 8a_5') = -30$
7. $(16a_4' + 64a_6' + z_1) = 90$
8. $(-2a_1' - 8a_2' - 2c_1 - z_1) = -30$
9. $(16a_4' + 64a_6') = 180.$

There are no other equations obtainable consistent with (2.23) and using Δ_1 and Δ_2 of the general form $(-\Box + X)$. We now have nine equations in thirteen unknowns. This is a much worse algebraic situation than before when we had 35 equations in 36 unknowns. Thus we see that when using (2.23) it seems necessary to consider the three cases given in § 3, even if one wants to solve only for $A_2(\Box^2 + A^{(\kappa\varepsilon)}\nabla_\kappa\nabla_\varepsilon + C^\kappa\nabla_\kappa + Z)$. Therefore § 3 constitutes a new derivation of Gilkey's (1980) results.

The recent work of Barvinsky and Vilkovisky (1985) obtains an expression for $A_2(\Box^2 + \Omega^{(\kappa\alpha\beta)}\nabla_\kappa\nabla_\alpha\nabla_\beta + A^{(\kappa\varepsilon)}\nabla_\kappa\nabla_\varepsilon + C^\kappa\nabla_\kappa + Z)$ which is the most general fourth-order operator with leading symbol given by some power of the metric. We note immediately (e.g. from the invariants made up of the Riemann tensor only) that their results are not consistent with (2.10). A further comparison between their results and our confirms this, and can be easily obtained in the limit of Riemann flat space (so that all covariant derivatives commute with each other). Setting $E^{(\kappa}g^{\alpha\beta)}$ equal to $\Omega^{(\kappa\alpha\beta)}$ in the operator above we can compare their expression with our (3.4). As the calculation would be a long and tedious one, we do not do this for their complete expression but instead present only a few of their terms rewritten. When we do this we obtain

$$\begin{aligned}180[A_2(\Box^2 + E^{(\kappa}g^{\alpha\beta)}\nabla_\kappa\nabla_\alpha\nabla_\beta + A^{(\kappa\varepsilon)}\nabla_k\nabla_\varepsilon + C^\kappa\nabla_\kappa + Z)]\\ = \{-15A^{\kappa\varepsilon}E_\kappa E_\varepsilon + 30A^{\kappa\varepsilon}E_{\kappa;\varepsilon} - \tfrac{15}{2}AE^\kappa E_\kappa\\ -30AE^\kappa{}_{;\kappa} + \tfrac{15}{2}A_{\kappa\varepsilon}A^{\kappa\varepsilon} + \tfrac{15}{4}A^2 + \ldots\}.\end{aligned} \tag{4.4}$$

Comparing with (3.4) we detect factor of $\frac{1}{2}$ discrepencies (as we did with the Riemann terms).

Let us now discuss this troublesome factor of 2. We believe, at least as far as loop corrections are concerned, that Barvinsky and Vilkovinsky (1985) and Fradkin and Tseytlin (1981, 1982, 1985) have, in introducing this factor in their algorithms, done nothing more than use a slightly different definition for the a_2 coefficient (in particular) than that of the standard works of Gilkey (1980), Hawking (1977) and Dowker and Critchley (1976) and references therein. That is, they use a definition such that, for some positive number k:

$$\bar{a}_{(1/2)d}(\Delta^k) = k\bar{a}_{(1/2)d}(\Delta) \tag{4.5}$$

where the bars indicate these are the non-standard coefficients. When property (4.5) holds it is also necessary to alter the standard definitions for the zeta function as well as others (the need for this does not seem to be discussed by Barvinsky and Vilkovinsky (1985) and Fradkin and Tseytlin (1981, 1982, 1985)). The new definition for the zeta function, for example, is easily found to be

$$\bar{\zeta}(s, \Delta) = \frac{2}{p}\sum_i \lambda_i^{-s} \qquad \text{for } p = 2v. \tag{4.6}$$

Only then does the analytically continued value for $\bar{\zeta}(0, \Delta)$ give results consistent with (4.5). This exactly parallels the analysis given in § 2. Relating the standard and non-standard definitions for the zeta function, we have

$$\bar{\zeta}(s, \Delta) = \frac{2}{p}\zeta(s, \Delta). \tag{4.7}$$

It is then a simple matter to use the integral representations for these two zeta functions (see (2.7)) together with (4.7) to relate the standard and non-standard asymptotic expansion coefficients. We obtain

$$\bar{A}_l(\Delta) = \frac{2}{p}A_l(\Delta). \tag{4.8}$$

This is exactly the relation we obtained in (2.18). Apparently, if done properly, using the non-standard definition amounts to nothing more than essentially putting in, and then taking out, an extraneous factor in a somewhat complicated way—at least as far as loop corrections are concerned. Thus we see that it seems consistency *can* be maintained dispite non-standard definitions (4.5) for the asymptotic expansion coefficients. However to do so requires related change in the standard definitions of the zeta function and others. We believe in the interest of simplicity, and reducing confusion with well established results and procedures already in the literature, it would be convenient for conventional definitions to be maintained. However relation (4.8) gives the simple relationship between the two asymptotic expansion coefficients, and can be used to transform the work of Barvinsky and Vilkovisky (1985) and Fradkin and Tseytlin (1981, 1982, 1985) into the standard form. Having said this, we now move on.

Although loop corrections *per se* are not the major concern in this paper, we make the following observation. By considering higher-order operators we discovered that in order to make (2.17) consistent with (2.10) relation (2.18) seemed necessary. Apparently, as they stand the divergent parts of ln det Δ given in (2.15) are not consistent with (2.10) for higher-order operators. Rather (2.15) is consistent with (4.5). Following

(2.18) we introduce an additional factor $p/2$ everywhere in (2.15) to obtain the coefficients of the divergent parts of $\ln \det \Delta$ in terms of asymptotic expansion coefficients consistent with (2.10). We obtain that parts diverging as $t^{(2l-d)/p}$ (for $(2l-d)<0$) have coefficient

$$-\frac{p^2}{2(2l-d)} A_l(\Delta) \tag{4.9a}$$

and the logarithmically divergent part has coefficient

$$-\tfrac{1}{2} p A_l(\Delta). \tag{4.9b}$$

The difference between (4.9) and (2.15) is a factor of $p/2$. Therefore for second-order operators these expressions give exactly the same coefficients of diverging parts. This is attractive in light of the fact that loop corrections for second-order operators are well established and have been calculated in many different ways, all giving consistent results. However, for higher-order operators the actor $p/2$ becomes non-trivial. It seems at first sight that (2.15) and (4.9) will yield different results. This however is not so due to the relation (4.8) between the coefficients A_l and $\bar{A}_l$. Let us consider a fourth-order gravity example to make this clear. Following Christensen (1982), the divergent part of the one-loop effective action is

$$W^{(1)}_{\text{div}} = \tfrac{1}{2}(\ln \det F_{ij} - \ln \det \gamma_{\alpha\beta} - 2 \ln \det F^{\alpha}{}_{\beta})_{\text{div}} \tag{4.10}$$

where F_{ij} is a fourth-order operator, and $\gamma_{\alpha\beta}$ and $F^{\alpha}{}_{\beta}$ are second-order operators given by Christensen (1982). In terms of asymptotic expansion coefficients satisfying the standard expression (2.10) we have, using (4.9), the coefficient of the logarithmically divergent parts of (4.10)

$$\tfrac{1}{2}(-2A_2(F_{ij}) + A_2(\gamma_{\alpha\beta}) + 2A_2(F^{\alpha}{}_{\beta})) \tag{4.11}$$

whereas for the non-standard asymptotic expansion coefficients satisfying (4.5), we use (2.15) and obtain

$$\tfrac{1}{2}(-\bar{A}_2(F_{ij}) + \bar{A}_2(\gamma_{\alpha\beta}) + 2\bar{A}_2(F^{\alpha}{}_{\beta})). \tag{4.12}$$

Although (4.10) and (4.11) seem to have non-trivial differences, they are in fact the same due to the relation (4.8) between the coefficients A_l and $\bar{A}_l$, and the fact that F_{ij} is a fourth-order operator, and $\gamma_{\alpha\beta}$ and $F^{\alpha}{}_{\beta}$ are both second-order operators. This example points to the care that must be exercised when considering higher-order operators, if extraneous factors are not to be inadvertently introduced.

Note Added. While this paper was being refereed an interesting article by Hae Won Lee and Pong Youl Pac (1986 *Phys. Rev.* D **33** 1012) appeared. We note that here as well, these authors' a_2 coefficients satisfy (4.5) rather than (2.10).

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References

Barth N H 1985 *Class. Quantum Grav.* **2** 497
—— 1987 *J. Phys. A: Math. Gen.* **20** 857
Barth N H and Christensen S M 1983 *Phys. Rev.* D **28** 1876
Barvinsky A O and Vilkovisky G A 1985 *Phys. Rep.* **119** 1
Christensen S M 1982 *Quantum Structure of Space and Time* ed M J Duff and C J Isham (Cambridge: Cambridge University Press)
DeWitt B 1965 *The Dynamical Theory of Groups and Fields* (New York: Gordon and Breach)
—— 1967 *Phys. Rev.* **162** 1239
Dowker J S and Critchley R 1976 *Phys. Rev.* D **13** 3224
Fradkin E S and Tseytlin A A 1981 *Phys. Lett.* **104B** 377
—— 1982 *Nucl. Phys.* B **201** 469
—— 1985 *Phys. Rep.* **119** 233
Gilkey P B 1974 *The Index Theorem and the Heat Equation* (Boston: Publish or Perish)
—— 1975 *Adv. Math.* **15** 344
—— 1980 *Duke Math. J.* **47** 511
Hawking S W 1977 *Commun. Math. Phys.* **55** 133
McKean H P Jr and Signer I M 1967 *J. Diff. Geom.* **1** 43
Pais A and Uhlenbeck G 1950 *Phys. Rev.* **79** 145
Stelle K S 1977 *Phys. Rev.* D **16** 953
Tomboulis E T 1985 *UCLA/85/TEP/11 preprint*
York J W 1979 *Sources of Gravitational Radiation* ed L Smarr (Cambridge: Cambridge University Press)